Section 3: Taniyama/Shimura

Elliptic curves are relatively simple equations, and their solutions (in the complex plane) form a surface of genus one. So they essentially make a doughnut shape. Simple. Modular forms on the other hand are infinitely symmetric and impossible to picture. On the surface (pun intended), these two advanced mathematical objects have nothing to do with each other, and nothing to do with Fermat's Last Theorem. So why are we studying them? Because as if often the case in mathematics, not everything is as it appears.

To begin to unravel the connection, we must go to post-war Japan. After World War II, Japan was in disarray and ruins. Many of the older generations were disheartened and discouraged. This attitude found its way into academia as well. University faculty in the late 1940's and early 1950's was uninspiring and tired. Many students left the country to find better opportunities and more enthusiasm elsewhere. But out of this environment came a pair of young mathematicians who would make an amazing claim.

Yutaka Taniyama and Goro Shimura were good friends and after graduating in the early 1950's they both stayed in Japan and took positions at the University of Tokyo. In an effort to spark interest in their field of mathematics, the two friends decided to organize a mathematical conference in 1955, the Tokyo-Nikko Symposium on Algebraic Number Theory. It was at this obscure conference halfway around the world where statements would be made that would have a profound impact on Fermat's Last Theorem, even if it would take 40 years for the rest of the world to understand the importance.

The proceedings of the conference included 36 problems, four of which were written by Taniyama. Taniyama was not a very careful mathematician, details were not his strength and errors were common in his work. But he was a gifted mathematician nevertheless. Shimura describes his good friend's abilities best:

> *Taniyama was not a very careful person as a mathematician. He made a lot of mistakes. But he made mistakes in good directions, so eventually he got to right answers. I tried to imitate him, but I found out that it is very difficult to make good mistakes.*

Embodied in the four problems of Taniyama was a remarkable conjecture, albeit in vague terms. Taniyama seemed to be implying that there was a connection between elliptic curves and modular forms. It wasn't stated very clearly, perhaps because Taniyama did not fully understand the connection himself, but the intuition was there. Any relationship between the automorphic functions of Poincaré and elliptic curves that went back to Diophantus certainly was not obvious. In the years that followed, many prominent mathematicians tried to decipher what Taniyama had in mind. Sadly, he was never able to provide his own complete explanation because in 1958 he took his own life. But his colleague and friend Shimura continued the work. Within a decade, he was able to clarify Taniyama's gut feeling and make a bold conjecture. It's called a "theorem" here, because it has since been proven. But we will get to that in Section 3.4.

Theorem 3.3.1 (Taniyama-Shimura) Every rational elliptic curve is a modular form.

This conjecture was unbelievable. In fact it wasn't even shocking or upsetting, because no one even thought it was possible. When André Weil discussed this with Taniyama prior to his death, he gave only passing consideration to the notion. Speaking of elliptic curves and modular forms, he said "They look completely different and mysterious." Years later, after Shimura clarified and made the bold claim, Jean-Pierre Serre approached him at a party and said "I don't think that your results on modular curves are any good. Why, they don't even apply to an arbitrary elliptic curve." Shimura replied "Such a curve is always modular." Apparently, this caused such disbelief in Serre that he went to Weil and relayed what Shimura had said. Weil later addressed Shimura "Did you really say that?" Yes, he did. The following years did not see much effort at proving Taniyama-Shimura, for many reasons. Could it possibly be true? How would someone even attempt to show it? What difference did it make? No one saw the connection to Fermat's Last Theorem until the 1980's.